A DISINTEGRATION THEOREM.

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Abstract. A new approach to disintegration of measures is presented, allowing one to drop the usually taken separability assumption. The main tool is a result on fibers in the spectrum of algebra of essentially bounded functions established recently by the first-named author.

In this note X, Y, Z will be compact spaces and the word "measurable" will concern their Borel sigma-filelds $\mathcal{B}_X$, $\mathcal{B}_Y$, $\mathcal{B}_Z$. Given a complex Borel measure μ on X and a measurable mapping $P : X \to Z$ we denote by $P(\mu)$ the *pushforward measure* defined on Z by

$$P(\mu)(E) := \mu(P^{-1}(E)), \quad E \in \mathcal{B}_Z,$$

so that

$$\int_Z h\, dP(\mu) = \int_X (h \circ P)\, d\mu, \quad h \in C(Z).$$

Now, given a continuous function $f \in C(X)$ we denote by

$$g_f := \frac{d(P(f\mu))}{d(P(|\mu|))},$$

the Radon-Nikodym derivative of the pushforward measures for " μ times density f" with respect to that of the variation measure $|\mu|$. Bearing in mind their absolute continuity we obtain for any $\psi \in L^1(P(|\mu|))$ the equalities

(1)
$$\int_X \psi(P(x))g_f(P(x))\, d|\mu|(x) = \int_Z \psi(z)g_f(z)\, dP(|\mu|)(z) = \int_X \psi(P(x))f(x)\, d\mu(x)$$

Of course $g_f \in L^1(P(|\mu|))$.

Lemma 1. *For any $f \in C(X)$ we have $g_f \in L^\infty(P|\mu|)$ and $\|g_f\|_\infty \leqslant \|f\|$.*

Proof. Let $h \in L^1(P(|\mu|))$. Then, as in (1), we get

$$\left|\int hg_f\, d(P(|\mu|))\right| = \left|\int h\, d(P(f\mu))\right| = \left|\int (h \circ P) f\, d\mu\right|$$
$$\leqslant \|f\| \int |h \circ P|\, d|\mu| = \|f\| \int |h|\, d(P(|\mu|)) = \|f\|\|h\|_1.$$

So g_f as a functional on $L^1(P(|\mu|))$ has norm estimated by $\|f\|$ (the sup-norm over X) and the result follows. $\square$

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Denote by η the measure $P(|\mu|)$ and assume that its total variation satisfies $\|\eta\| = 1$ and that g_f is real. (The general case will easily follow by splitting into the real and imaginary parts and multiplying by a constant). Let Y be the spectrum of the Banach algebra $L^\infty(\eta)$. It is a totally disconnected compact space with its Gel'fand topology.

There is a canonical projection $\Pi : Y \to Z$ (cf.[2],[3]) that assigns to a multiplicative linear functional $y \in Y$ a unique point $\Pi_y \in Z$ so that for any $f \in C(Z)$ one has $f(\Pi_y) = y([f])$ with $[f] \in L^\infty(\eta)$ denoting the coset modulo equality $[\eta]$ a.e. The measure η lifts to a Borel measure $\tilde{\eta}$ on Y so that that $\Pi(\tilde{\eta}) = \eta$. As follows from [2],[3], such a Borel measure on Y is actually unique. The main result in [3] provides for arbitrarily chosen $h \in L^\infty(\eta)$ (here $h = g_f$) a dense open set $U = U_h$ in Y, having full measure $\tilde{\nu}$ and such that $\widehat{h}$ is constant on each set $\Pi^{-1}(\{z\}) \cap U_h$ for $z \in Z$.

For $z \in Z$ denote

$$\mathcal{U}_z := \{\Pi^{-1}(\Pi(V)) : V \subset Y, V \text{ closed-open},\ z \in \Pi(V)\}.$$

For any $z \in Z$ we define a linear functional $\Phi_z : C(X) \to \mathbb{R}$ putting

$$\Phi_z(f) := \operatorname*{Lim}_{E \in \mathcal{U}_z} \frac{1}{\tilde{\eta}(E)} \int_E \widehat{g_f}\, d\tilde{\eta}. \tag{2}$$

Here Lim denotes a Banach limit. We require it to be only linear and located between the lower- and upper limits with respect to the directed family $\mathcal{U}_z$. By Lemma 1, Φ_z is bounded, of norm less or equal 1. Hence for each $z \in Z$ there exists a complex Borel measure ν_z on X such that

$$\Phi_z(f) = \int f\, d\nu_z \quad \text{for} \quad f \in C(X), \quad \|\nu_z\| \leqslant 1 \quad \text{for} \quad z \in Z. \tag{3}$$

Lemma 2. *For $a \in \Pi^{-1}(\{z\}) \cap U_{g_f}$ we have $\Phi_z(f) = \widehat{g_f}(a)$.*

Proof. Let $a \in \Pi^{-1}(\{z\}) \cap U$, where $U = U_{g_f}$. For an arbitrary $\varepsilon > 0$ take a cosed-open neighbourhood V_ϵ of a such that $|\widehat{g_f}(y) - \widehat{g_f}(a)| < \varepsilon$ for $y \in V_\varepsilon$ and put $E_\varepsilon := \Pi^{-1}(\Pi(V_\varepsilon))$. This is possible since clopen sets form a base of topology for Y (cf.[2]). Since $\widehat{g_f}$ is constant on each fiber intersected with U we also have $|\widehat{g_f}(y) - \widehat{g_f}(a)| < \varepsilon$ for $y \in E_\varepsilon \cap U$. But as we have $\tilde{\eta}(Y \setminus U) = 0$, the integral means over the sets E and $E \cap U$ are equal (for $d\tilde{\eta}$). The above estimate by ε for $\widehat{g_f} - \widehat{g_f}(a)$ yields the same bound ε for the differences between the integral means over any $E \in \mathcal{U}_z$ such that $E \subset E_\varepsilon$. Passing to the Banach limits, we get

$$|\Phi_z(f) - \widehat{g_f}(a)| \leqslant \varepsilon. \tag{4}$$

Since ε was arbitrary we get $\Phi_z(f) = \widehat{g_f}(a)$. □

Let us recall that for a family of measures ν_z, $z \in Z$ the vector-valued integral $\int_Z \nu_z\, d\eta$ is the measure μ_1 such that for any continuous function h on X we have

$$\int h d\mu_1 = \int_Z \Big(\int h(x) d\nu_z(x) \Big)\, d\nu(z). \tag{5}$$

The disintegration of a Borel probability measure μ on a compact space X with respect to a mapping $P : X \to Z$ is a measurable family of probability measures ν_z satisfying (5) with $\mu_1 = \mu$. The existence of disintegration under certain assumptions including the separability of X is shown in [1]. If one considers probability measures μ, for constant function $f_0 = 1$ one has $g_{f_0} = 1$ and $\Phi_z(f_0) = 1$, hence our measures ν_z obtained in (3) are probabilistic. For complex measures μ the integral representation (5) still has its meaning and we may call it the disintegration of μ in this general case.

We are now in position to state our main result

Theorem 3. *The family of measures ν_z, $z \in Z$ forms a disintegration of the measure μ with respect to P.*

Proof. Taking $\psi = 1$ in (1), using (3) we get for $f \in C(X)$, $U = U_{g_f}$ the equalities

$$\begin{aligned}\int_X f\,d\mu = \int_Z g_f\,d\eta = \int_Y \widehat{g}_f\,d\tilde{\eta} = \int_{Y\cap U} \widehat{g}_f(a)\,d\tilde{\eta}(a) \\ = \int_{Y\cap U} \Phi_{\Pi(a)}(f)\,d\tilde{\eta}(a) = \int_Y \Phi_{\Pi(a)}(f)\,d\tilde{\eta}(a) \\ = \int_Z \Phi_z(f)\,d\eta(z) = \int_Z \Big(\int f\,d\nu_z\Big)\,d\eta(z) = \int_Z \Big(\int f\,d\nu_z\Big)\,d(P(|\mu|))(z).\end{aligned}$$

□

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